Substitution on infinite alphabets and generalized Bratteli-Vershik models

Shrey Sanadhya

University of Iowa

Joint work with Sergii Bezuglyi and Palle Jorgensen

March 10, 2021

Plan for this talk

Cantor d.s   Borel d.s

- Introduction to Bratteli diagrams and generalized Bratteli diagrams
- Two versions of Kakutani-Rokhlin tower construction for a class of substitution dynamical systems on a countably infinite alphabets (known as left determined substitution).
- Bratteli-Vershik (B-V) models for such substitution dynamical systems.
- Using the Bratteli-Vershik model we find explicit expressions for invariant and ergodic measures for such substitution dynamical system.
- Example. ✓

Preliminaries

- Let $\mathcal{A}$ be a countably infinite set, called an *alphabet*. We denote by $\mathcal{A}^{\mathbb{Z}}$ the bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ on $\mathcal{A}$.
- Note that $\mathcal{A}^{\mathbb{Z}}$ is a non-compact Polish space.
- For $x \in \mathcal{A}^{\mathbb{Z}}$ we denote by $\mathcal{L}_n(x)$, the set of all words of length n in x.
- *Language* of x is defined by $\mathcal{L}(x) := \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(x)$.
- A *substitution* σ on $\mathcal{A}$ is a map from $\mathcal{A}$ to $\mathcal{A}^+$ (the set of finite non-empty words on $\mathcal{A}$), which associates to the letter $a \in \mathcal{A}$ the word $\sigma(a) \in \mathcal{A}^+$, with length $h_a := |\sigma(a)| < \infty$.
- We define *language of a substitution* σ by : $\mathcal{L}_\sigma = \{\text{factors of } \sigma^n(a) : \text{for some } n \geq 0, a \in \mathcal{A}\}$.
- Define $X_\sigma = \{x \in \mathcal{A}^{\mathbb{Z}} : \mathcal{L}(x) \subset \mathcal{L}_\sigma\} \subset \mathcal{A}^{\mathbb{Z}}$.

Preliminaries cont.

- The *left shift* $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, is defined by $(Tx)_k = x_{k+1}$, for all $k \in \mathbb{Z}$.
- X_σ is a Polish space and is closed under T. We call (X_σ, T) the subshift associate with substitution σ.
- (X_σ, T) is Borel dynamical system i.e. T is a homeomorphism of Polish space.
- For a finite string $\bar{x} = (x_0, ..., x_n)$ of length n, denote by $[\bar{x}]$ a *cylinder set* : $[\bar{x}] := \{y = (y_i) \in X_\sigma : y_0 = x_0, ..., y_n = x_n\}$.
- We say that σ is of bounded length if there exists an integer $C \geq 2$ such that for every $a \in \mathcal{A}$, $|\sigma(a)| \leq C$. $n = 1$ $\sigma(1)$
- By identifying $\mathcal{A}$ with $\mathbb{Z}$, we say that σ is of bounded size, if it is of bounded length and there exists a positive integer t (independent of n and minimal possible), such that for every $n \in \mathbb{Z}$, if $m \in \sigma(n)$, then $m \in \{n - t, ..., n, ..., n + t\}$. $2t+1$

Left determined substitution on infinite alphabet

Definition ([Ferenczi 2006]). We say that a substitution σ on a countable alphabet $\mathcal{A}$ is left determined if there exists $N \in \mathbb{N}$ such that, any word $w \in \mathcal{L}_\sigma$ of length at least N, has a unique decomposition $w = w_1.....w_s$, such that each $w_i = \sigma(a_i)$ for unique $a_i \in \mathcal{A}$, except that w_1 may be a suffix of $\sigma(a_1)$ and w_s may be a prefix of $\sigma(a_s)$.

$|w| = N$

$$w = \overbrace{\underbrace{b_1\, b_2\, b_3}_{\text{suffix of } \sigma(a_1)}}^{w_1} \; \overbrace{\underbrace{b_4\, b_5\, b_6\, b_7}_{=\sigma(a_2)}}^{w_2} \; \ldots\ldots \; \overbrace{\underbrace{b_{N-4}\, b_{N-3}\, b_{N-2}}_{=\sigma(a_{s-1})}}^{w_{s-1}} \; \overbrace{\underbrace{b_{N-1}\, b_N}_{\text{prefix of } \sigma(a_s)}}^{w_s}$$

can can

Left determined substitution on infinite alphabet

Definition ([Ferenczi 2006]). We say that a substitution σ on a countable alphabet $\mathcal{A}$ is left determined if there exists $N \in \mathbb{N}$ such that, any word $w \in \mathcal{L}_\sigma$ of length at least N, has a unique decomposition $w = w_1.....w_s$, such that each $w_i = \sigma(a_i)$ for unique $a_i \in \mathcal{A}$, except that w_1 may be a suffix of $\sigma(a_1)$ and w_s may be a prefix of $\sigma(a_s)$.

$$w = \overbrace{\underbrace{b_1\, b_2\, b_3}_{\substack{\text{suffix of}\\ \sigma(a_1)}}}^{w_1}\ \overbrace{\underbrace{b_4\, b_5\, b_6\, b_7}_{=\sigma(a_2)}}^{w_2}\ \ldots\ldots\ \overbrace{\underbrace{b_{N-4}\, b_{N-3}\, b_{N-2}}_{=\sigma(a_{s-1})}}^{w_{s-1}}\ \overbrace{\underbrace{b_{N-1}\, b_N}_{\substack{\text{prefix of}\\ \sigma(a_s)}}}^{w_s}$$

Example: *The squared drunken man substitution*:

$$n \mapsto (n-2)\, n\, n\, (n+2)\, ;\ n \in 2\mathbb{Z}$$

is left determined (see [F06]).

Example: a (non-simple, finite rank) Bratteli diagram

$r, s : E \to V$

$\forall v \in V \setminus V_0 \quad r^{-1}(v) \neq \emptyset$

$\forall v \in V \quad s^{-1}(v) \neq \emptyset$

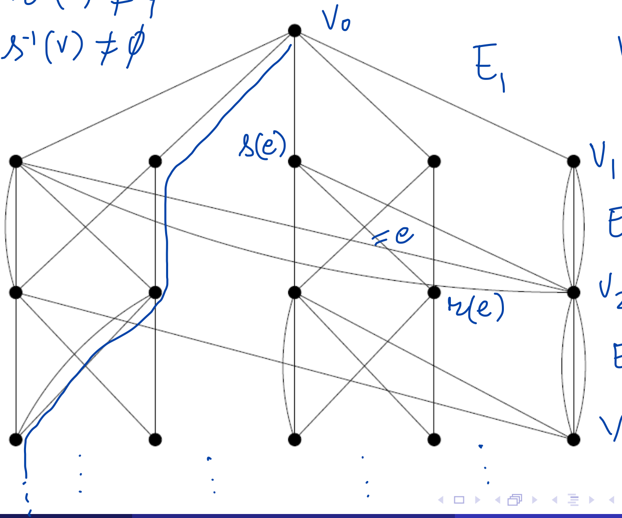
$B(V,E)$

$|V_i| < \infty$

$|E_i| < \infty$

$V = \bigcup_i V_i$

$E = \bigcup_i E_i$



$(e_i)_i$

$r(e_i) = s(e_{i+1}) \quad \forall i$

X_B

$\downarrow$

Path space

Definition of a Bratteli diagram

Definition

A Bratteli diagram is a graded infinite graph $B = (V, E)$ with the vertex set $V = \bigsqcup_{i \geq 0} V_i$ and edge set $E = \bigsqcup_{i \geq 1} E_i$:
1) $V_0 = \{v_0\}$ is a single point;

2) V_i and E_i are finite sets for every i;

3) edges E_i connect V_{i-1} to V_i: there exist maps r (range) and s (source) from E to V such that $r(E_i) \subseteq V_i, s(E_i) \subseteq V_{i-1}$, and $s^{-1}(v) \neq \emptyset$; $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

- B is stationary if it repeats itself below the first level.
- B is of finite rank if for all $n \geq 1$, $|V_n| \leq k$ for some positive integer k.
- We say a finite rank diagram B has rank d if d is the smallest integer such that $|V_n| = d$ infinitely often.

Definition of a Bratteli diagram (cont.)

The incidence matrix F_n is a $|V_n| \times |V_{n-1}|$ matrix with entries

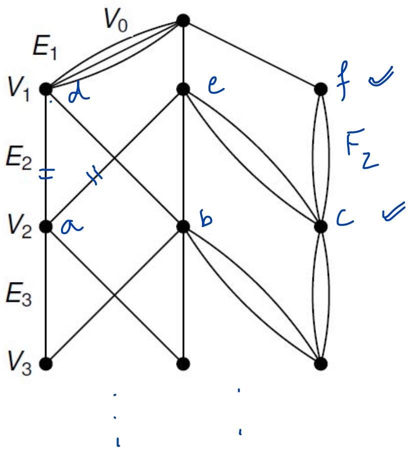
$$f_{v,w}^{(n)} = |\{e \in E_n : s(e) = w, r(e) = v\}|, \ \ v \in V_n, w \in V_{n-1}.$$

A Bratteli diagram is called simple if $\forall n \ \exists m > n$ such that $F_m \cdots F_{n+1} > 0$ (all entries are positive).

A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a finite or infinite path. Let X_B be the set of infinite paths starting at the top vertex v_0. Then X_B a 0-dimensional compact metric space w.r.t. the topology generated by cylinder sets

$$[\overline{e}] := \{x \in X_B : x_i = e_i, \ i = 0, \dots, n\}.$$

Incidence matrix (Example)



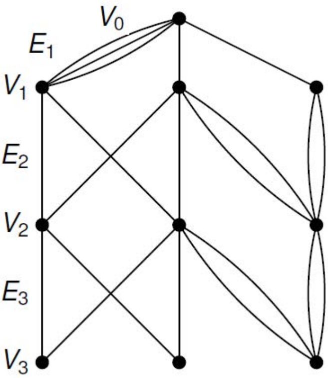
The diagram is *stationary* with incidence matrix

$$F = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \end{array}$$

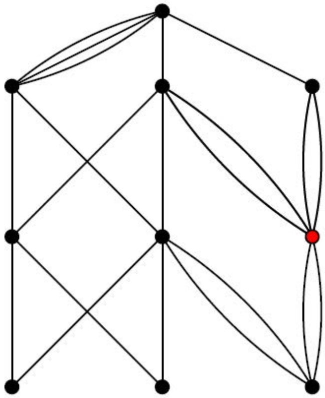
The sequence (F_n) of incidence matrices determine the structure of a Bratteli diagram.

Topology on the path space X_B: two paths are close if they agree on a large initial segment.

Ordered Bratteli diagrams

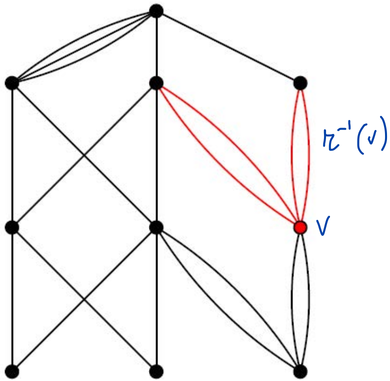


Ordered Bratteli diagrams



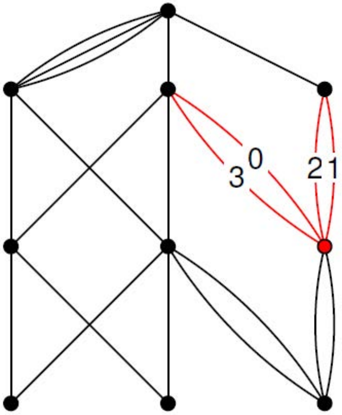
- Take a vertex $v \in V \setminus V_0$.

Ordered Bratteli diagrams



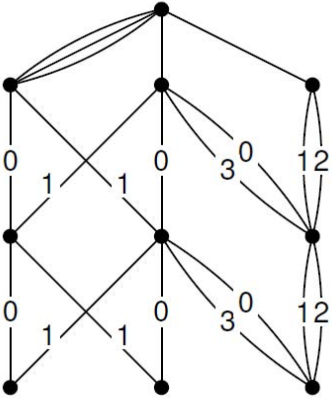
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.

Ordered Bratteli diagrams



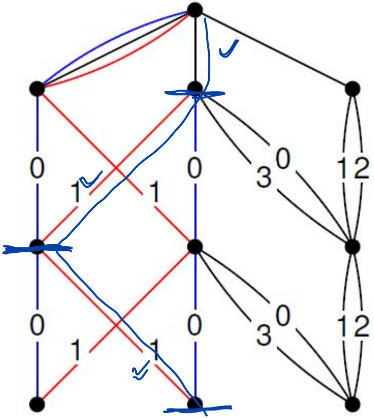
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.
- Enumerate edges from $r^{-1}(v)$

Ordered Bratteli diagrams



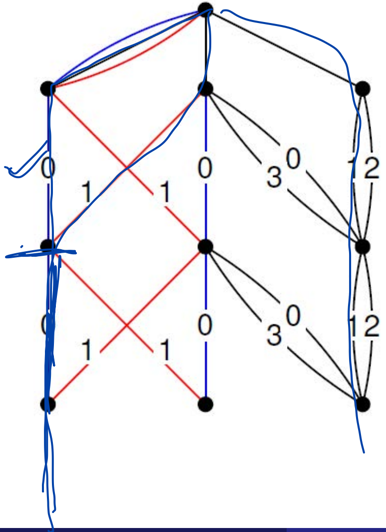
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.
- Enumerate edges from $r^{-1}(v)$
- Do the same for every vertex.

Ordered Bratteli diagrams



- An infinite path $x = (x_n)$ is called **maximal** if x_n is *maximal* in $r^{-1}(r(x_n))$. Similarly, **minimal** paths are defined.

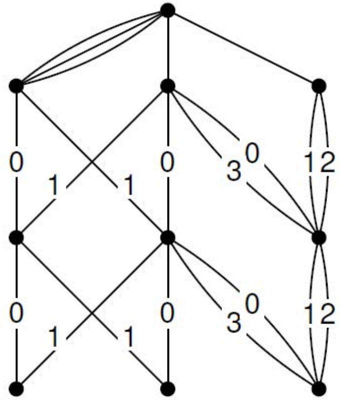
Ordered Bratteli diagrams



- An infinite path $x = (x_n)$ is called **maximal** if x_n is *maximal* in $r^{-1}(r(x_n))$. Similarly, **minimal** paths are defined.
- The sets $X_{\max}$ and $X_{\min}$ of all maximal and minimal paths are non-empty and closed.

Ordered Bratteli diagrams

Vershik map

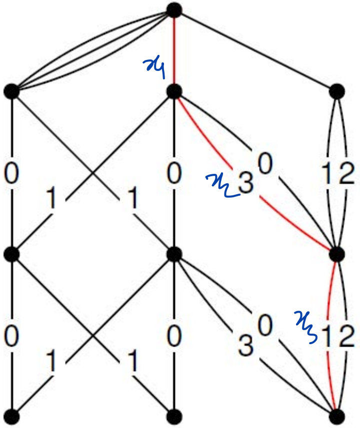


Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Ordered Bratteli diagrams

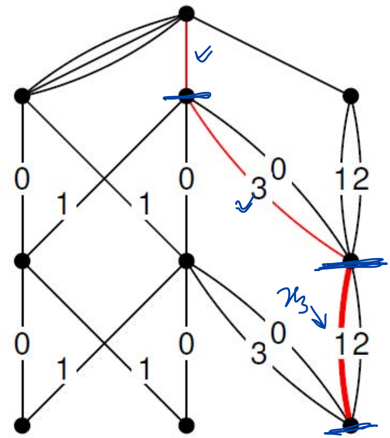
Vershik map



Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.



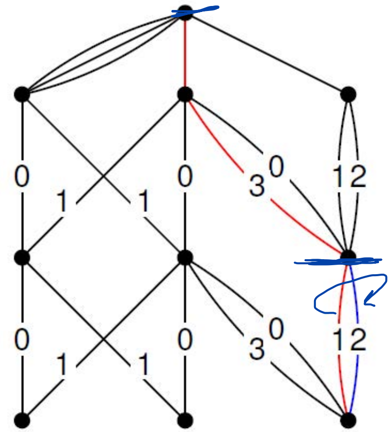
Define the Vershik map

$\varphi_B : X_B \setminus X_{max} \to X_B \setminus X_{min}$:

Fix $x \in X_B \setminus X_{max}$.

Find the first k with non-maximal x_k.

$$k = 3$$



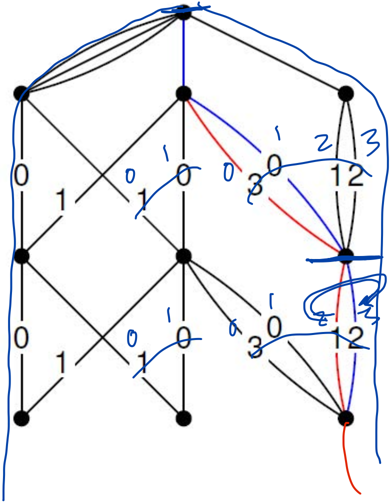
Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with non-maximal x_k.

Take x_k to its successor $\overline{x}_k$.



Define the Vershik map

$\varphi_B : \underline{X_B \setminus X_{\max}} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with non-maximal x_k.

Take x_k to its successor $\overline{x}_k$.

Connect $s(\overline{x}_k)$ to the top vertex V_0 by the minimal path.

φ_B (red) = blue

Ordered Bratteli diagrams

Vershik map

- φ_B is defined everywhere on $X_B \setminus X_{\max}$
- $\varphi_B(X_B \setminus X_{\max}) = X_B \setminus X_{\min}$

φ_B

Definition

If the map φ_B can be extended to a homeomorphism of X_B such that $\varphi_B(X_{\max}) = X_{\min}$, then (X_B, φ_B) is called a Bratteli-Vershik system and φ_B is called the Vershik map.

Question:

Under what conditions on a Bratteli diagram does the Vershik map exist?

Ordered Bratteli diagrams

Vershik map

- φ_B is defined everywhere on $X_B \setminus X_{\max}$
- $\varphi_B(X_B \setminus X_{\max}) = X_B \setminus X_{\min}$

Definition

If the map φ_B can be extended to a homeomorphism of X_B such that $\varphi_B(X_{\max}) = X_{\min}$, then (X_B, φ_B) is called a Bratteli-Vershik system and φ_B is called the Vershik map.

Question:

Under what conditions on a Bratteli diagram does the Vershik map exist?

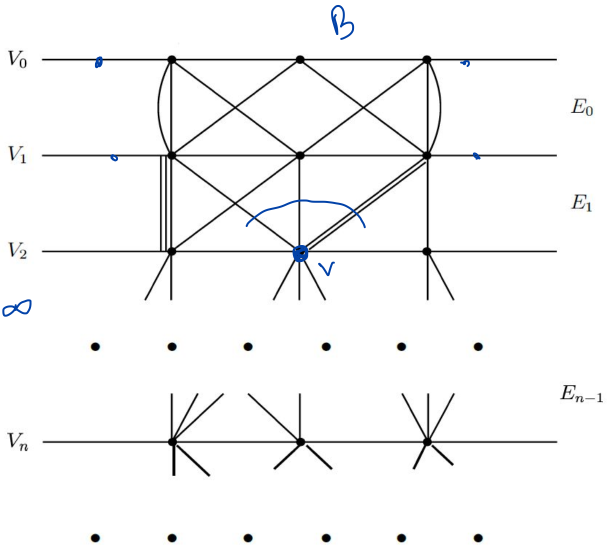
Answer:

If a Bratteli diagram is simple, then the Vershik map **always** exists (e.g., use the left-to-right order).

Collection of results (partial list)

- Herman, Putnam, and Skau [HPS'92] showed that for every minimal Cantor dynamical system (X, T), there exists a simple, ordered Bratteli diagram B such that the corresponding Vershik map φ_B is conjugate to T.
- Bezuglyi, Dooley and Medynets (2005), Medynets (2006) extended above result to **aperiodic** Cantor dynamical systems.
- Giordano, Putnam, and Skau (1995) classified all minimal homeomorphisms of Cantor set with respect to **orbit equivalence.** By [HPS'92] it suffices to classify Vershik maps.
- Forrest (1997), Durand, Host, Skau (1999) described completely the class of dynamical systems that are represented by simple stationary Bratteli diagram. These are **minimal substitution dynamical systems**.

Generalized Bratteli diagrams example



$\mathcal{V} = $

$|V_i|$

$= $ ctbl infin.

$|E_i|$

$|r^{-1}(v)| < \infty$

Y_B

Generalized Bratteli diagrams (GBD)

Generalized Bratteli diagram $B = (V, E)$ is a countable graded graph $B = (V, E)$ with $V = \bigsqcup_{i \geq 0} V_i$ and $E = \bigsqcup_{i \geq 0} E_i$ such that,

(i) The set V_i, for $i \geq 0$ is countably infinite (identified with $\mathbb{Z}$). E_i is the set of edges between the levels V_i and V_{i+1};

(ii) range map r and source map s from E to V such that $r(E_i) \subset V_i$, $s(E_i) \subset V_{i-1}$, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \setminus V_0$;

(iii) for every $v \in V \setminus V_0$, the set $r^{-1}(v)$ is finite. For every $w \in V_i$, $v \in V_{i+1}$, the set of edges (denoted $E(w, v)$) between w and v is finite (or empty);

(iv) Put $|E(w, v)| = f^i_{vw}$. This defines a sequence of infinite *incidence matrices* $(F_n; n \in \mathbb{N}_0)$ whose entries are non-negative integers:

$$F_i = (f^{(i)}_{vw} : v \in V_{i+1}, w \in V_i), \quad f^{(i)}_{vw} \in \mathbb{N}_0.$$

Generalized Bratteli diagram (GBD) cont.

- If $F_n = F$ for all $n \in \mathbb{N}_0$, then diagram is called *stationary*.
- We denote by Y_B the set of infinite paths in $B = (V, E)$.
- Y_B is a Polish space using topology generated by cylinder sets $[\overline{e}] := \{x \in Y_B : x_i = e_i, \ i = 0, \ldots, n\}$.
- $B = (V, E)$ together with a linear order on $r^{-1}(v)$ for every $v \in V \setminus V_0$, is called an ordered generalized Bratteli diagram denoted by $B = (V, E, \geq)$.
- For an ordered diagram $B = (V, E, \geq)$, we define Vershik map $\varphi : Y_B \to Y_B$ as before.
- (Y_B, φ) is a Borel dynamical system.

Std Borel space

Theorem (Bezuglyi, Dooley, Kwiatkowski (2006))

Let T be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Then there exists an ordered generalized Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi_B : Y_B \to Y_B$ such that (X, T) is isomorphic to (Y_B, φ_B).

Back to infinite substitution : Kakutani-Rokhlin towers

For $a_i \in \mathcal{A}$; $i = \{1, ..., n\}$, we denote by $[a_1 ... a_n]$ a cylinder set of length n.

Theorem 1. Let σ be a *left determined substitution* on countably infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then for any $n \in \mathbb{N}$, we have a partition of X_σ into K-R towers

$$X_\sigma = \bigsqcup_{a_1....a_n \in \mathcal{L}_n(\sigma)} \bigsqcup_{k=0}^{h_1+...+h_n-1} T^k[\sigma(a_1....a_n)]$$

where $h_k = |\sigma a_k|$ for $k \in \{1, .., n\}$.

Back to infinite substitution : Kakutani-Rokhlin towers

For $a_i \in \mathcal{A}$; $i = \{1, ..., n\}$, we denote by $[a_1 ... a_n]$ a cylinder set of length n.

Theorem 1. Let σ be a *left determined substitution* on countably infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then for any $n \in \mathbb{N}$, we have a partition of X_σ into K-R towers

$$X_\sigma = \bigsqcup_{a_1 \ldots . a_n \in \mathcal{L}_n(\sigma)} \bigsqcup_{k=0}^{h_1 + \ldots + h_n - 1} T^k[\sigma(a_1 \ldots . a_n)]$$

where $h_k = |\sigma a_k|$ for $k \in \{1, .., n\}$.

Theorem 2. Let σ be *left determined* and bounded length substitution on a countable infinite alphabet $\mathcal{A}$, and (X_σ, T) be the corresponding subshift. Then for every $n \in \mathbb{N}$, we have a partition of X_σ into K-R towers

$$X_\sigma = \bigsqcup_{a_i \in \mathcal{A}} \bigsqcup_{k=0}^{h_i^n - 1} T^k[\sigma^n a_i] \text{ , where } |\sigma a_i| = h_i.$$

Construction of Bratteli-Vershik model

We used *Theorem 1* and *Theorem 2* to obtain:

Corollary 3. Let σ be a left determined substitution of bounded size on countably infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then there exists two sequence (A_n) and (B_n) of Borel sets with $A_0 = B_0 = X_\sigma$ and for $n > 0$,

$$A_n = \bigsqcup_{a_i \in \mathcal{A}} [\sigma^n a_i], \text{ and } B_n = \bigsqcup_{a_1 \dots a_n \in \mathcal{L}_n(\sigma)} [\sigma(a_1 \dots a_n)].$$

such that

(a) $X_\sigma = A_0 \supset A_1 \supset A_2 \supset A_3 \dots$ and $X_\sigma = B_0 \supset B_1 \supset B_2 \supset B_3 \dots$.
(b) Both $\bigcap_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} B_n$ are countably infinite.
(c) A_n and B_n are complete T-sections for each $n \in \mathbb{N}$.
(d) For each $n \in \mathbb{N}$ every point in A_n and B_n is recurrent.

Bratteli-Vershik models for substitution on infinite alphabet

Using *Corollary 1*, (sets B_n) we obtain Bratteli-Vershik model for left determined substitution of bounded size on countably infinite alphabet.

Theorem 4. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then there exists an ordered generalized–Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi : Y_B \to Y_B$ such that (X_σ, T) is isomorphic to (Y_B, φ).

Bratteli-Vershik models for substitution on infinite alphabet

Using *Corollary 1*, (sets B_n) we obtain Bratteli-Vershik model for left determined substitution of bounded size on countably infinite alphabet.

Theorem 4. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then there exists an ordered generalized–Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi : Y_B \to Y_B$ such that (X_σ, T) is isomorphic to (Y_B, φ).

Using *Corollary 1*, (sets A_n) we obtain stationary Bratteli-Vershik model.

Theorem 5. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Then there exists a stationary ordered generalized–Bratteli diagram $\tilde{B} = (\tilde{V}, \tilde{E}, \geq)$ and a Vershik map $\tilde{\varphi} : Y_{\tilde{B}} \to Y_{\tilde{B}}$ such that (X_σ, T) is isomorphic to $(Y_{\tilde{B}}, \tilde{\varphi})$.

Perron-Frobenius theorem for countable matrix

Theorem 6. (Generalized PF Theorem, see [K 98]). Suppose F is a countable, non-negative, irreducible, aperiodic matrix. Suppose that F is recurrent. Then there exists a Perron-Frobenius eigenvalue $\lambda = \lim_{n\to\infty}(f_{ij}^{(n)})^{\frac{1}{n}} > 0$ such that:

(a) there exist unique strictly positive left ℓ and right r eigenvectors corresponding to λ,

(b) $\ell \cdot r = \sum_i \ell_i r_i < \infty$ if and only if F is positive recurrent.

Tail-invariant measure on stationary GBD

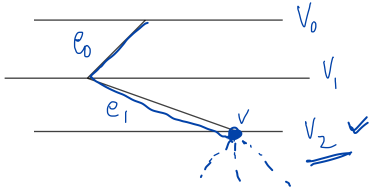
Theorem 7. Let $B = B(F)$ be a stationary generalized-Bratteli diagram such that the incidence matrix F, is irreducible, aperiodic and recurrent. Then there exists a tail invariant measure μ on the path space Y_B.

(1) Let $[\bar{e}] = [e_0 e_1 \ldots . e_{n-1}]$ denote a cylinder set of length n such that $r(\bar{e}) = v \in V_n$, then we define :

$$\mu([\bar{e}]) = \frac{\ell_v}{\lambda^n}.$$

where ℓ is the left eigenvector corresponding to Perron value λ of F.

Example



$$\mu([e_0 e_1]) = \frac{\ell_v}{\lambda^{2}}$$

Tail-invariant measure on stationary GBD

Theorem 7. Let $B = B(F)$ be a stationary generalized-Bratteli diagram such that the incidence matrix F, is irreducible, aperiodic and recurrent. Then there exists a tail invariant measure μ on the path space Y_B.

(1) Let $[\bar{e}] = [e_0 e_1 \ldots . e_{n-1}]$ denote a cylinder set of length n such that $r(\bar{e}) = v \in V_n$, then we define :

$$\mu([\bar{e}]) = \frac{\ell_v}{\lambda^n}.$$

where ℓ is the left eigenvector corresponding to Perron value λ of F.

(2) The measure μ is finite if and only if the left eigenvector $\ell = (\ell_v)$ has the property $\sum_v \ell_v < \infty$.

From tail-invariant to shift-invariant measure

Since dynamical systems (X_σ, T) and (Y_B, φ) are isomorphic, we can push the tail-invariant measure to shift-invariant measure.

Corollary 8. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ and (X_σ, T) be the corresponding subshift. Assume that the countably infinite substitution matrix M is irreducible, aperiodic, and recurrent.

(1) Then there exists a shift-invariant measure ν on X_σ.

(2) Let ℓ be the left eigenvector of M corresponding to the Perron value λ of M. The measure ν is finite if and only if the left eigenvector $\ell = (\ell_i)$ has the property $\sum_i \ell_i < \infty$.

Theorem 9. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ such that the substitution matrix M is irreducible, aperiodic, and recurrent. Then the shift-invariant probability measure ν (defined in Corollary 7) on X_σ is ergodic.

Proof Sketch. We work with the stationary Bratteli-Vershik model $(B, \varphi, \geq)$ of (X_σ, T). Since μ is probability, we have $\sum_v \ell_v = 1$.

M is irreducible : For all i, j there exists some n such that $m_{ij}^{(n)} > 0$.
Moreover, for a fixed state i there exists k such that $m_{ii}^{(n)} > 0$ for all $n \geq k$.

Ergodic shift-invariant probability measure

Theorem 9. Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ such that the substitution matrix M is irreducible, aperiodic, and recurrent. Then the shift-invariant probability measure ν (defined in Corollary 7) on X_σ is ergodic.

Proof Sketch. We work with the stationary Bratteli-Vershik model $(B, \varphi, \geq)$ of (X_σ, T). Since μ is probability, we have $\sum_v \ell_v = 1$.

M is irreducible : For all i, j there exists some n such that $m_{ij}^{(n)} > 0$.

Moreover, for a fixed state i there exists k such that $m_{ii}^{(n)} > 0$ for all $n \geq k$.

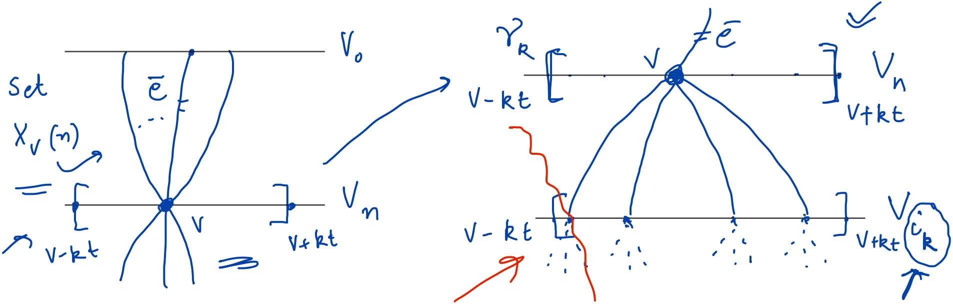
Let $[\bar{e}]$ be a cylinder set such that $r(\bar{e}) = v \in V_n$. We will show that $\mu(\mathcal{R}([\bar{e}])) = 1$, where $\mathcal{R}$ denotes the tail-equivalence relation.

Proof sketch cont.

For $n > 0$ and $v \in V_n$ we define, $X_v(n) = \{x \in Y_B : r(x_{n-1}) = v \in V_n\}$.

For a fixed $k \in \mathbb{N}$ and t as in the definition of bounded size the set $\mathcal{V}_k = \{v - kt, ..., v + kt\}$ denotes a set of $2kt + 1$ vertices.

Since M is irreducible, there exists a level (say V_{i_k}, $i_k > n$) such that v is connected to every $u \in \mathcal{V}$ on level V_{i_k}.



Proof sketch cont.

For $n > 0$ and $v \in V_n$ we define, $X_v(n) = \{x \in Y_B : r(x_{n-1}) = v \in V_n\}$.

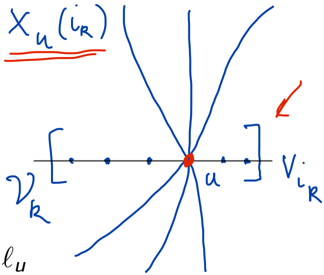
For a fixed $k \in \mathbb{N}$ and t as in the definition of bounded size the set $\mathcal{V}_k = \{v - kt, ..., v + kt\}$ denotes a set of $2kt + 1$ vertices.

Since M is irreducible, there exists a level (say V_{i_k}, $i_k > n$) such that v is connected to every $u \in \mathcal{V}$ on level V_{i_k}.

Consider the set $\overline{U_k} = \bigcup_{u \in \mathcal{V}_k} X_u(i_k) \subset \mathcal{R}([\bar{e}])$.

Note that $\mu(\overline{U_k}) = \sum_{u \in \mathcal{V}_k} \frac{|E(V_0, u)|}{\lambda^{i_k}} \ell_u$.

$$\mu(\overline{U_k}) = \sum_{u \in \mathcal{V}_k} \frac{\sum_{w=u-i_k t}^{u+i_k t} f_{uw}^{(i_k)}}{\lambda^{i_k}} \ell_u$$



Proof sketch cont.

By switching the order of summations

$\ell . f^{i_k}$

$= \ell . \lambda$

$$\mu(\overline{U_k}) = \sum_{w=u-i_k t}^{u+i_k t} \frac{\sum_{u\in\mathcal{V}_k} (\ell_u) \cdot (f_{uw}^{(i_k)})}{\lambda^{i_k}}$$

For $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\sum_{u\in\mathcal{V}_k} (\ell_u) \cdot (f_{uw}^{(i_k)}) > \lambda^{i_k}\ell_w - \epsilon$$

Hence, $\mu(\overline{U_k}) > \sum_{w=u-i_k t}^{u+i_k t} \frac{\lambda^{i_k}\ell_w - \epsilon}{\lambda^{i_k}} = \sum_{w=u-i_k t}^{u+i_k t} \ell_w - \sum_{w=u-i_k t}^{u+i_k t} \frac{\epsilon}{\lambda^{i_k}}$

Proof sketch cont.

$$\mu(\overline{U_k}) > \sum_{w=u-i_k t}^{u+i_k t} \ell_w - \frac{\epsilon}{\lambda^{i_k}}(2i_k t + 1),$$

$k \to \infty$, $i_k \to \infty$, observe that

$$\sum_{w=u-i_k t}^{u+i_k t} \ell_w \longrightarrow \sum_{w \in \mathbb{Z}} \ell_w = 1 \text{ and } \frac{(2i_k t + 1)}{\lambda^{i_k}} \longrightarrow 0,$$

Thus $\mu(\overline{U_k}) \longrightarrow 1$. Since $\mu(\overline{U_k}) \subset \mathcal{R}([\bar{e}])$, we get $\mu(\mathcal{R}([\bar{e}])) = 1$.

Thus for any two open sets $\mathcal{O}_1, \mathcal{O}_2 \in Y_B$, there exists $n \in \mathbb{Z}$ such that $\mu(\varphi_B^n(\mathcal{O}_1) \cap \mathcal{O}_2) > 0 \implies$

For any two Borel sets A_1, A_2, there exists $n \in \mathbb{Z}$ such that $\mu(\varphi_B^n(A_1) \cap A_2) > 0 \implies \mu$ is ergodic. $\square$

Summary of ongoing work

Let σ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ such that the substitution matrix M, is irreducible and recurrent, then following could be said about ~~the then~~ the shift-invariant measure ν defined earlier

Substitution matrix	$\sum_v \ell_v$	Shift-invariant measure		
		Type	**Ergodic**	**Unique**
Positive recurrent	Finite	Finite	Yes	Yes BET
Null recurrent	Finite	Finite	Yes	??
Positive recurrent	Infinite	σ-finite	??	If it is ergodic then it is unique. Hofs
Null recurrent	Infinite	σ-finite	??	??

Example

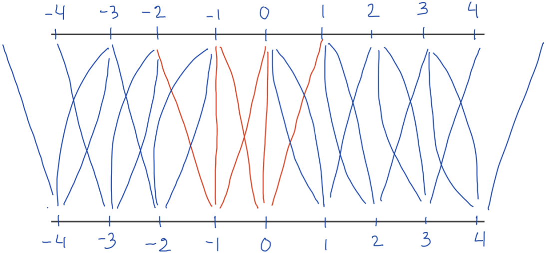
One step fwd, two step back substitution on $\mathbb{Z}$. Define σ by

$$\mathbf{-1} \mapsto -2 -1 \;\; 0 \;\; ; \;\; \mathbf{0} \mapsto -1 \;\; 0 \; 1$$

$$n \mapsto (n-1)(n+1)(n+1) \; ; \;\; n \leq -2$$

$$n \mapsto (n-1)(n-1)(n+1) \; ; \;\; n \geq 1$$

This substitution is left determined, with irreducible, positive recurrent matrix; $\lambda = 3$; left e.v, $\ell = (\ldots \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \ldots ..)$



[F06] Sébastien Ferenczi. Substitution dynamical systems on infinite alphabets. Ann. Inst. Fourier (Grenoble), 56:2315-2343, 2006.

[K 98] Bruce P. Kitchens. Symbolic Dynamics. One-sided, Two-sided and Countable State Markov Shifts. Springer Universitext. ISBN 3-540-62738-3. 1998.

Thank You

Addendum

Some definitions from theory of Stochastic Process

- Countably infinite non-negative matrix F is called *irreducible* if for all $i, j \in \mathbb{Z}$ there exists some $n \in \mathbb{N}$ such that $f_{ij}^{(n)} > 0$.
- An irreducible matrix F has period p if, for all vertices $i \in \mathbb{Z}$, $p = \gcd\{\ell : f_{ii}^{(\ell)} > 0\}$. If $p = 1$, the matrix F is called *aperiodic*.
- An irreducible aperiodic matrix F admits a *Perron-Frobenius eigenvalue* λ, defined by, $\lambda = \lim_{n\to\infty}(f_{ii}^{(n)})^{\frac{1}{n}}$ (independent of i).
- An irreducible aperiodic matrix F is called *transient* if $\sum_n f_{ij}^{(n)}\lambda^{-n} < \infty$; otherwise, F is called *recurrent*.
- For a recurrent matrix F, define $\ell_{ij}(1) = f_{ij}$ and $\ell_{ij}(n+1) = \sum_{k\neq i} \ell_{ik}(n) f_{kj}$.
- The matrix F is called *null-recurrent* if $\sum_n n\ell_{ii}(n)\lambda^{-n} < \infty$; otherwise F is called *positive recurrent*.